Soluție

1.a. $\det A = \begin{vmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{vmatrix} = -a^3$.

b. $A^2 = \begin{pmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}$ apoi se verifică ușor că $A^2 X = XA^2$.

c. $aI_3 + bA = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 0 & ba \\ 0 & ba & 0 \\ ba & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & ba \\ 0 & a+ba & 0 \\ ba & 0 & a \end{pmatrix}$. Notăm cu $B = aI_3 + bA$.

$$A \cdot B = \begin{pmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & ba \\ 0 & a+ba & 0 \\ ba & 0 & a \end{pmatrix} = \begin{pmatrix} ba^2 & 0 & a^2 \\ 0 & a^2+ba^2 & 0 \\ a^2 & 0 & ba^2 \end{pmatrix}$$

$B \cdot A = \begin{pmatrix} a & 0 & ba \\ 0 & a+ba & 0 \\ ba & 0 & a \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{pmatrix} = \begin{pmatrix} ba^2 & 0 & a^2 \\ 0 & a^2+ba^2 & 0 \\ a^2 & 0 & ba^2 \end{pmatrix}$ deci matricea $aI_3 + bA \in G$.

2.a. Avem $f(-1) = 1^{1004} - 1 = 0$.

b. Punând $x = 1$ obținem : $f(1) = 3^{1004} + 1 = a_0 + a_1 + a_2 + ... + a_{2009}$ de unde $a_0 + a_1 + a_2 + ... + a_{2009}$ este un număr par.

c. $r(X) = \dfrac{3^{1004}+1}{2} \cdot X + \dfrac{3^{1004}+1}{2}$.