

Rezolvare

1.a. $f'(x) = -\frac{2}{x^3} - \frac{2}{(x+1)^3}, \forall x > 0.$

b. $f'(x) = -\frac{2}{x^3} - \frac{2}{(x+1)^3} < 0, \forall x > 0$, deci f este descrescătoare pe $(0, +\infty)$.

c. $\lim_{x \rightarrow \infty} x^3 f'(x) = \lim_{x \rightarrow \infty} x^3 \left(-\frac{2}{x^3} - \frac{2}{(x+1)^3} \right) = \lim_{x \rightarrow \infty} \left(-2 - \frac{2x^3}{(x+1)^3} \right) = -4.$

2.a. $\int_1^e \left(f(x) - \frac{\ln x}{x} \right) dx = \int_e^e x dx = \frac{x^2}{2} \Big|_1^e = \frac{e^2 - 1}{2}.$

b. $\int_1^e f(x) dx = \int_1^e \left(\frac{\ln x}{x} + x \right) dx = \left(\frac{\ln^2 x}{2} + \frac{x^2}{2} \right) \Big|_1^e = \frac{e^2}{2}.$

c. $I_n = \left(\frac{1}{2} \ln^2 x \right) \Big|_{e^n}^{e^{n+1}} = \frac{2n+1}{2}$, obținem $I_{n+1} - I_n = 1 \Rightarrow$ progresie aritmetică cu rația 1.