Soluție

1.a) $f'(x)=\left(x^2\ln x\right)'=2x\ln x+x^2\cdot\frac{1}{x}=x(2\ln x+1),\ \forall x>0$.

b) $\lim\limits_{x\to+\infty}\frac{f'(x)}{x\ln x}\overset{\text{cf. pct.a)}}{=}\lim\limits_{x\to+\infty}\frac{x(2\ln x+1)}{x\ln x}=\lim\limits_{x\to+\infty}\frac{2\ln x+1}{\ln x}=\lim\limits_{x\to+\infty}\left(2+\frac{1}{\ln x}\right)=2$.

c) $f'(x)=0\Rightarrow x(2\ln x+1)=0\Rightarrow \ln x=-\frac{1}{2}\Rightarrow x=e^{-\frac{1}{2}}\in(0,+\infty)$. Din tabelul de variație rezultă că $A\left(e^{-\frac{1}{2}},-\frac{1}{2e}\right)$ este punctul de minim al funcției f. Deci $f(x)\geq-\frac{1}{2e}$, oricare ar fi $x>0$.

2. a) Avem $\int_0^1 f(x)e^{-x}\,dx=\int_0^1 xe^x\cdot e^{-x}\,dx=\int_0^1 x\,dx=\left.\frac{x^2}{2}\right|_0^1=\frac{1}{2}$.

b) $f'(x)=\left(xe^x\right)'=(x+1)e^x,\ \forall x\in\mathbb{R}$. $\int_0^1 f''(x)\,dx=\int_0^1\left(f'(x)\right)'dx=\left.f'(x)\right|_0^1=f'(1)-f'(0)=2e-1$.

c) Avem $\int_1^2\frac{f\left(x^2\right)}{x}\,dx=\int_1^2 xe^{x^2}\,dx\underset{u'(x)=2x}{\overset{u(x)=x^2}{=}}\frac{1}{2}\int_1^2 u'(x)\cdot e^{u(x)}\,dx=\left.\frac{1}{2}e^{u(x)}\right|_1^2=\frac{e\left(e^3-1\right)}{2}$.