Soluție

1.a) $f'(x)=\left(\frac{1}{x}-\frac{1}{x+1}\right)'=\frac{1}{(x+1)^2}-\frac{1}{x^2}, \forall x>0$.

b) $f'(x)\frac{-2x-1}{x(x+1)}\le 0,\ \forall x\in(0;\infty)\Rightarrow f$ este descrescătoare pe $(0;\infty)$.

Dacă $x\in(1;\infty)$, $\sqrt{x}<x\Rightarrow f\left(\sqrt{x}\right)\ge f(x)\Leftrightarrow \frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x}+1}\ge f(x)$, oricare ar fi $x\in(1;+\infty)$.

c) $\lim\limits_{x\to+\infty} x f(x) f\left(\frac{1}{x}\right)=\lim\limits_{x\to+\infty}\left(x\cdot\frac{1}{x(x+1)}\cdot\frac{1}{\frac{1}{x}\left(\frac{1}{x}+1\right)}\right)=\lim\limits_{x\to+\infty}\frac{x^2}{(x+1)^2}=1$.

2. a) $I_0+I_2=\int_1^{\sqrt{3}}\left(\frac{1}{x^2+1}+\frac{1}{x^2\left(x^2+1\right)}\right)dx=\int_1^{\sqrt{3}}\frac{1}{x^2}dx=-\frac{1}{x}\Big|_1^{\sqrt{3}}=\frac{\sqrt{3}-1}{\sqrt{3}}$.

b) $I_1=\int_1^{\sqrt{3}}\frac{1}{x\left(x^2+1\right)}dx=\int_1^{\sqrt{3}}\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx=\frac{1}{2}\ln\frac{3}{2}$.

c) $I_n+I_{n-2}=\int_1^{\sqrt{3}}\left(\frac{1}{x^n\left(x^2+1\right)}+\frac{1}{x^{n-2}\left(x^2+1\right)}\right)dx=\int_1^{\sqrt{3}}\frac{1}{x^n}dx=\frac{x^{-n+1}}{-n+1}\Big|_1^{\sqrt{3}}=\frac{1}{n-1}\left(1-\frac{1}{\left(\sqrt{3}\right)^{n-1}}\right), \forall n\in\mathbb{N}, n\ge 2$.