Soluție

1.a) într-adevăr, *u* este o funcție de două ori derivabilă, $u'(x)=e^x(\sin x+\cos x)$, deci $u'(0)=1$ și $f(0)=0$.

b) Avem $\left(1+f(x)\right)^{\frac{1}{x}}=\left[\left(1+f(x)\right)^{\frac{1}{f(x)}}\right]^{\frac{f(x)}{x}}$ și $\lim\limits_{x\to 0}\left(1+f(x)\right)^{\frac{1}{f(x)}}=e$, deoarece $\lim\limits_{x\to 0}f(x)=f(0)=0$.

Apoi $\lim\limits_{x\to 0}\frac{f(x)}{x}=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=1$.

c) $\lim\limits_{x\to 0}\frac{f^n(x)-x^n}{x^{n+1}}=\lim\limits_{x\to 0}\frac{f(x)-x}{x^2}\cdot\left(\frac{f^{n-1}(x)}{x^{n-1}}+\frac{xf^{n-2}(x)}{x^{n-1}}+\frac{x^{n-2}f(x)}{x^{n-1}}+\frac{x^{n-1}}{x^{n-1}}\right)$, iar

$\lim\limits_{x\to 0}\frac{f(x)-x}{x^2}=\lim\limits_{x\to 0}\frac{f'(x)-1}{2x}=\lim\limits_{x\to 0}\frac{1}{2}\frac{f'(x)-f'(0)}{x-0}=\frac{f''(0)}{2}$,

$\lim\limits_{x\to 0}\frac{f^k(x)}{x^k}=\lim\limits_{x\to 0}\left(\frac{f(x)-f(0)}{x-0}\right)^k=\left(f'(0)\right)^k=1,\ \forall k\in\{0,1,2,...,n-1\}$.

2. a) $g(x)=\int\limits_0^x f(t)\,dt=\int\limits_0^x\frac{1}{1+t}\,dt=\ln(1+x)$.

b) $\int_0^1 f^2(x)g(x)dx=\int_0^1\frac{\ln(1+x)}{(x+1)^2}dx=-\int_0^1\left(\frac{1}{x+1}\right)'\ln(1+x)\,dx=-\frac{1}{2}\ln 2+\int_0^1\frac{1}{(x+1)^2}dx=\frac{1-\ln 2}{2}$.

c) Pentru $k\in\{0,1,2,...,n-1\}$ cu teorema de medie $\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)dx=\left(\frac{k}{n}-\frac{k-1}{n}\right)f(c_k)$, cu $c_k\in\left[\frac{k-1}{n},\frac{k}{n}\right]$.

Apoi, deoarece *f* este descrescătoare, $f(c_k)\ge f\left(\frac{k}{n}\right)$. Rezultă $\frac{1}{n}\sum\limits_{k=1}^{n}f\left(\frac{k}{n}\right)\le\sum\limits_{k=1}^{n}\int\limits_{\frac{k-1}{n}}^{\frac{k}{n}}f(x)dx=\int\limits_0^1 f(x)dx=\ln 2$.